

Quantum correlations and synchronization measures

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The phenomenon of spontaneous synchronization is universal and only recently advances have been made in the quantum domain. Being synchronization a kind of temporal correlation among systems, it is interesting to understand its connection with other measures of quantum correlations. We review here what is known in the field, putting emphasis on measures and indicators of synchronization which have been proposed in the literature, and comparing their validity for different dynamical systems, highlighting when they give similar insights and when they seem to fail.

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I. INTRODUCTION

Synchronization phenomena [1, 2] and quantum correlations [3–5] have been studied for a long time by two different communities, and only recently their relation started to be explored. The common ingredient for the emergence of both features is the mutual interaction between the components of a system, and in the quantum regime the potential relation between synchronization and the presence of mutual information, discord, entanglement or other correlations has been recently explored.

The phenomenon of spontaneous or mutual synchronization refers to the ability of two or more systems, that would display different dynamics when separate, to evolve coherently when coupled. In the case of oscillatory dynamics this corresponds to achieving oscillation at a common frequency. This concept has been further refined and generalized in chaotic systems to encompass several scenarios such as, for instance, lag synchronization, generalized synchronization, or phase synchronization [6]. In general, the definition of classical synchronization itself refers to some similarity in the time evolutions, i.e. some temporal *correlation* between the local dynamical variables of the involved systems. Therefore, this is a definition associated to classical trajectories. The counterpart, and eventually generalization, in the quantum regime can follow different approaches.

The first works on the subject of quantum synchronization were actually dealing with entrainment, where an external driving acts as a pacemaker, in systems such as spin-boson with modulated driving [7], driven resonator with one [8] or two superconducting qubits [9], and more recently driven quantum Van der Pol oscillators [10, 11]. In the case of entrainment, or forced synchronization, the driver is a strong external field, generally classical, and is not influenced by the interaction with the system. Different is the case of mutual synchronization that refers to the emergence of synchronization as a collective phenomenon, leading to a coherent dynamics out of different coupled units, in the absence of a driver.

Mutual synchronization has been recently predicted for spins interacting with a common bath [12] and for the average positions of quantum optomechanical systems [13, 14]. The first analysis in the quantum regime dealt with harmonic networks [15, 16] looking at quantum noise synchronization and showing its counterpart in relation with classical and quantum correlations, showing the same trend for mutual information and quantum discord. Synchronization in the dynamics of second-order quadratures (squeezing) was considered there with an exact approach and role of local, global and independent baths was elucidated. In Ref.[17] it was instead considered the question about the limits to perfect synchronization imposed by quantum fluctuations and uncertainty relations. The phenomenon is characterized with a synchronization error and is discussed in the

context of coupled optomechanical devices. Also in Ref.[10] there is a discussion of coherently coupled Van der Pol oscillators characterizing their phase-locking in phase space.

When extending the concepts of synchronization into the quantum regime, a first question is about what defines this phenomenon, as actually in the classical regimes it refers to the dynamics of phase space trajectories and, in general, classical variables. This chapter reviews this question showing different approaches as well as the specific peculiarities reported so far for quantum synchronization with respect to the classical case. In the following section we give a brief overview of the platforms where quantum synchronization is under study and then review the characterizations and measures proposed for this phenomenon. We then discuss some general questions and possible future directions.

II. SYNCHRONIZATION IN QUANTUM SYSTEMS

The phenomenon of synchronization is paradigmatic in a large variety of biological, physical, and chemical systems, operating in the classical regime, as reviewed for instance in [1, 2, 6, 18, 19]. Some fascinating examples are fireflies flashing at once, cardiac myocytes acting as pacemakers, or the swaying motion of the millennium bridge due to the crowd walk in synchrony. The first reported observation of classical synchronization was described as “sympathy of two clocks” and dates back to XVII century, when Huygens observed pendula hanging on a wall in a boat (see extracts and references in [1]). A reproduction of Huygens’ original drawing is presented in Fig.1. An equivalent popular experiment is with metronomes on the same bar placed on two cylinders (cans) free to roll [20]. The extension to chaotic systems has also been a wide field of research, establishing the possibility to observe this phenomenon in spite of the high sensitivity to small differences in the initial conditions or device parameters.

When moving to microscopic systems, synchronization phenomena are expected to take place, and several recent works address quantum synchronization in nanomechanical devices, harmonic oscillators and spin systems.

A. Nanomechanical devices

Optomechanical devices exploit radiation pressure to couple coherently light and matter motional degrees of freedom allowing to explore different aspects of synchronization in a flexible, hybrid and highly sensitive platform, where operation in the quantum regime has been achieved [22]. Spontaneous synchronization of optomechanical devices has been predicted theoretically considering mechanical coupling [13] as well as coupling through a common optical mode [14], focusing on the average positions and laying the base for the study of quantum signatures of synchronization in these devices. Phase-coherent mechanical oscillations have been shown in regular optomechanical crystals considering the effects of quantum noise [23].

Reported experiments with microdisks [24] and arrays [21] and nanomechanical resonators interacting through an optical racetrack [25] display synchronization of the average (classical) motional degree of freedom. In Fig. 1 the device used in Ref. [21] is reproduced. The possibility to lock distant optomechanical oscillators has also been explored in the classical regime [26, 27]. Recently it was also reported the experimental realization of spontaneous synchronization among micro- [28] and nano-electromechanical [29] autonomous oscillators. Quantum signatures of synchronization phenomena in experiments on optomechanical devices have not yet been reported.



FIG. 1. A historical perspective of synchronization. Left panel: Original drawing of Huygens of two synchronizing pendulum clocks attached to a common support. Middle panel: Swarms of fireflies illuminate the undergrowth in a forest (photo by Kei Nomiyama/ Barcroft Media). Right panel: Micromechanical oscillator arrays coupled through light (figure taken from Ref. [21]).

B. Linear and non-linear oscillators

Among theoretical models, both linear and non-linear oscillators have been considered theoretically in the quantum regime. Van der Pol oscillators have been investigated in the quantum regime [10, 11, 30, 31] and in comparison with the classical one [1]. These models exhibit self-sustained oscillations and spontaneous synchronization due to coherent [10, 32] and dissipative coupling [30, 31], as well as phase locking [10] and frequency entrainment [11] in presence of external drive. The realization of Van der Pol oscillators in physical platforms operating in the quantum regime has been suggested in trapped ions [10] and optomechanical oscillators [31].

Self-sustained oscillators, like Van der Pol oscillators are a well-known platform for studying synchronization. However, due to their non-linearity the analysis in the quantum regime can be only addressed in limited cases and under various approximations, such as truncation of the Wigner function, linearization around stable states [33] or in the limit of infinite non-linear couplings favoring few low-energy Fock states [10]. An *exact* analysis can be performed in linear systems like harmonic networks; these have been considered in order to identify the conditions for quantum synchronization beyond approximations and to clarify the role of losses, comparing diffusive and reactive couplings in Refs. [15, 16, 34]. The analysis of networks in squeezed vacuum [16] shows that under dissipation in separate equivalent baths (a common assumption), independently on the strength of the coupling the oscillators will not be able to synchronize, while in any other more complex dissipation scenarios (common bath, local bath, etc...), the presence of one less damped normal mode of the system allows for a transient or asymptotic synchronization. By accessing few oscillators parameters this synchronization in the squeezing dynamics can be tuned in the network or in clusters.

C. Spin models

As mentioned before, quantum spin synchronization was first discussed under the perspective of entrainment to an external driving force, either in the general spin-boson framework [7] or considering superconducting two-level systems [8, 9]. An experimental observation was recently reported considering a damped current-biased Josephson junction [35].

Studies of spontaneous synchronization of spins within abstract theoretical models were performed in Refs. [12, 36] considering spin-boson dissipation. In Ref. [12], it was first observed that synchronization is induced by the coherent exchange of bath excitations between the two spins, while in Ref. [36] it was shown that pure dephasing is not able to generate synchronization. The formation of Chimera states was discussed in [37] considering an extended spin chain described by a non-Hermitian Hamiltonian. In Ref. [32], the authors analyzed the behavior of two qubits placed inside two coupled cavities where only the first one is driven by a laser. The steady-state synchronization of ensembles of dissipative, driven two-level atoms collectively coupled to a cavity mode, was studied in [38] and, under more general conditions in [39], where the authors also provided a direct analogy with the synchronization of classical phase oscillators. Following the experimental results of Ref. [40], where a self-rephasing mechanism was observed on the ground state of magnetically trapped ultracold atoms, synchronization within a full quantum model for the case of two non-dissipative, interacting macro-spins, was studied in Ref. [41].

Finally, a platform to probe synchronization was introduced in Ref. [42]. There, the authors considered two cold ions in microtraps and studied the synchronization between their motional degrees of freedom. The presence of synchronization was witnessed by the correlations developed by the electronic, discrete, degrees of freedom of the two ions.

D. Applications

Synchronization is clearly a resource in biological systems [1, 2]. The synchronized flashing of fireflies is a strategy so that the female can identify her species-specific flashing signal (Fig. 1). Synchronization of neuronal activity by phase locking of self-generated network oscillations dynamics is one of the coordinating mechanisms of the brain, and abnormalities in this process are at the basis of several diseases and dysfunctions, like epilepsy.

The achievement of a coherent dynamics out of different components (also due to experimental imperfections) is clearly a resource also in physical systems. An interesting application, for instance, is in cryptographic protocols based on chaotic carriers of signals [43]. In general, synchronization allows for enhancing of frequency stability, coherence and power output. Therefore applications are envisaged for precise frequency sources, time-keeping, and sensing [21, 29] and can be taken also to the domain of quantum technologies.

An application of a synchronization transition was recently proposed as an effective tool to probe the dynamics of a quantum system dissipating through a thermal bath [44]. Indeed, coupling the system to an external, detuned object (which plays the role of the probe) a transition between in-phase and anti-phase system-probe synchronization is observed as a function of the detuning and of the spectral density of the bath. Clearly, this transition can be observed monitoring the dynamics of the probe alone. Then, measuring the critical detuning at which the transition takes place amounts to getting information about the whole dissipative process.

III. MEASURES OF MUTUAL QUANTUM SYNCHRONIZATION

Synchronization refers to some coherence in the temporal dynamics of coupled systems and several measures are known in the classical realm [1, 2, 6]. Synchronization in presence of driving, entrainment, is typically encoded in the phase locking of the slave system with respect to the drive: the detuning between the slave oscillation and the driving frequency is typically plotted as a function of the frequency of the driver to identify the region of locking (zero detuning) and when this region's width is considered for different driving strengths one gets the synchronization region known as Arnold tongue [1].

In autonomous systems, synchronization can arise as a mutual phenomenon, the final dynamics coming from the interaction between components. The equivalent to an Arnold tongue appears by considering the relative coupling and detuning of the system components. Despite its intuitive conceptual definition, the quantification of synchronization in the quantum realm is a challenging problem where both temporal and quantum correlations come into play. Two or more objects, irrespective of their quantum or classical nature, do spontaneously synchronize if they adjust their own local dynamics to a common pace determined by their mutual interaction. Then, a good synchronization measure is expected to be able to capture this adjustment of rhythms, that can only be detected monitoring the behavior of all local units. Synchronization can be inferred observing how similar the local density matrices are, according to some meaningful criteria, or considering local observables and looking at their correlations in time. Furthermore, it is sometimes possible to deduce the behavior of local observables inspecting overall quantities, like, for instance, emission spectra.

A broad plethora of studies of synchronization in classical systems in the last three decades provides useful hints about possible approaches when moving into the quantum regime. The main difference with respect to classical systems is clearly that synchronization there generally refers to time trajectories and limit cycles in the phase space not valid in the quantum regime. On the other hand, classical synchronization has been already generalized in presence of noise and of chaos [6] where it is identified by assessing the 'similarity' of local dynamical evolutions quantified by several indicators. Indeed a number of these indicators can be taken into the quantum regime providing insightful approaches to quantum synchronization, as it is the case of the Pearson function and synchronization error introduced below.

These considerations do not exclude that manifestations of synchronization can be found also in global indicators, including mutual information and correlations, and that can be associated to genuine quantum properties of the whole state. Overall, when addressing the question of the identification of genuine quantum synchronization phenomena, two main approaches can be distinguished: one is to define synchronization in local observables and look for the presence of quantum correlations triggered by this phenomenon; an alternative approach is to define synchronization itself as a form of quantum correlation. In the following we give an overview of different physical cases where synchronization is expected to come out, discussing the interplay between local indicators and collective ones.

A. Pearson factor

The Pearson's correlation coefficient is a widely used measure of the degree of linear dependence between two variables. Calling X and Y the variables, it is defined as

$$\mathcal{C}_{X,Y} = \frac{E[XY] - E[X]E[Y]}{\sqrt{E[X^2] - E[X]^2} \sqrt{E[Y^2] - E[Y]^2}}, \quad (1)$$

where $E[\cdot]$ is an average value. As a consequence of the definition, $\mathcal{C}_{X,Y}$ gives a value between +1 and -1, where 1 indicates total positive correlation, 0 is the absence of correlation, and -1 is total negative correlation. Considering two functions $f(t)$ and $g(t)$ evolving in time, the Pearson's coefficient $\mathcal{C}_{f(t),g(t)}(t)$ can be calculated over a sliding window of length Δt replacing the expectation values with time averages: in this case

$$E[f](t, \Delta t) \equiv \overline{f(t, \Delta t)} = \frac{1}{\Delta t} \int_t^{t+\Delta t} f(t) dt. \quad (2)$$

Given two time-dependent variables A_1 and A_2

$$\mathcal{C}_{A_1,A_2}(t|\Delta t) = \frac{\int_t^{t+\Delta t} (A_1 - \bar{A}_1)(A_2 - \bar{A}_2) dt}{\sqrt{\int_t^{t+\Delta t} (A_1 - \bar{A}_1)^2 dt \int_t^{t+\Delta t} (A_2 - \bar{A}_2)^2 dt}}, \quad (3)$$

where

$$\bar{A}_i = \frac{1}{\Delta t} \int_t^{t+\Delta t} A_i dt. \quad (4)$$

By definition, this measure quantifies the temporal correlation between two classical trajectories and has been widely used in classical synchronization problems [1, 6].

In the quantum framework the trajectories A_i can be the expectation values of quantum operators, as moments at different orders of local observables, like $\langle \hat{N}_i \rangle$, $\langle \hat{x}_i^2 \rangle$, $\langle \hat{x}_i^4 \rangle$, σ_i^x This measure was first adopted in the framework of quantum synchronization in Ref. [15], using the quantum-mechanical expectation values of the second moments of the positions and momenta of two linearly coupled harmonic oscillators dissipating in a common environment. In this way the synchronization in the dynamics of second-order quadratures was captured. The same quantification of synchronization was also carried out in the case of an extended network of linear harmonic oscillators [16].

In the case of dissipating spin pairs, the Pearson's coefficient was adopted in Refs. [36, 44] to quantify the degree of synchronization between $\langle \sigma_1^x \rangle$ and $\langle \sigma_2^x \rangle$. A slightly different version of it, especially tailored to detect phase synchronization, was also analyzed in Ref. [32]. A qualitative analysis of the similarity between the time evolution of local averages of spin operators, even though without any explicit reference to the Pearson's measure, was also invoked by Orth *et al.* [12].

In general, this measure can be applied to any quantum problem when looking at temporal dynamics of local observables. The main advantages are (i) that it depends on the quantum signatures of the system (e.g. quantum noise, when going beyond first order moments) and (ii) that this measure has absolute reference values: reaching the maximum (minimum) value $C_{X,Y} = 1(-1)$ for perfect (anti-)synchronization.

B. Synchronization error

The averaged distance between classical trajectories has been largely used to study synchronization of chaotic systems, see e.g. the example of a pair of bidirectionally coupled Lorenz systems in Ref. [6] of two coupled chaotic systems. This synchronization error was first considered in [17] to study quantum synchronization of coupled optomechanical oscillators attaining limit cycles. For continuous variable (CV) systems the synchronization error reads

$$\mathcal{S}_c(t) = \langle q_-(t)^2 + p_-(t)^2 \rangle^{-1}, \quad (5)$$

where $q_- = (q_1 - q_2)/\sqrt{2}$ is the difference in position, and the same for momentum, of the objects of interest. At variance with the classical case where the average distance can go to zero, in the quantum domain this measure is bounded. It achieves a maximum value when the two quantum objects are synchronized, and is upper-bounded by the uncertainty principle

$$\mathcal{S}_c(t) \leq \frac{1}{2\sqrt{\langle q_-(t)^2 \rangle \langle p_-(t)^2 \rangle}} \leq 1. \quad (6)$$

A poor value of this quantity can come from two possible origins: either the mean value (first moment) of q_- and p_- is big, or because the variances of these operators are big. In order to neglect the first cause, it is also interesting to define a modified measure with

$$q_-(t) \rightarrow q_-(t) - \langle q_-(t) \rangle, \quad p_-(t) \rightarrow p_-(t) - \langle p_-(t) \rangle, \quad (7)$$

which is preferable if we want to study purely quantum effects.

While synchronization error in classical systems is generally addressed between the time series of two deterministic variables, as for example $q_1(t)$ and $q_2(t)$, the intrinsic probabilistic nature in the quantum domain enlarge this scenario. As for the Pearson factor, when comparing two operators $\hat{q}_1(t)$ and $\hat{q}_2(t)$, the corresponding first moments can behave independently of the second moments or moments of higher order. This is particularly the case for Brownian oscillators initialized in vacuum squeezed states. The intention of the authors in [17] is to be able to compare the two operators by introducing a quadratic error measure, as reported in optomechanical settings (see IV B section), gauging well (as can be seen from comparison to other synchronization indicators) both the synchronization of first moments and second moments. The relation of this measure with the synchronization of local dynamics is still an open question.

In a similar spirit a measure of phase synchronization is also introduced in [17], by writing the operator $a_j(t) := [q_j(t) + ip_j(t)]/\sqrt{2}$ of the j th system in the following way

$$a_j(t) = [r_j(t) + a'_j(t)]e^{i\phi_j(t)}, \quad (8)$$

where r_j and ϕ_j are the amplitude and phase of the expectation value of $a_j(t)$: $\langle a_j(t) \rangle = r_j(t)e^{i\phi_j(t)}$. Now the hermitian and anti-hermitian parts of $a'_j(t) := [q'_j(t) + ip'_j(t)]/\sqrt{2}$ can be interpreted as amplitude and phase fluctuations, and we can say that whenever $\langle a_1(t) \rangle$ and $\langle a_2(t) \rangle$ are phase locked we can define a phase shift with respect to this locking by the operator $p'_-(t) = [p'_1(t) - p'_2(t)]/\sqrt{2}$. Hence, a measure of phase synchronization is

$$\mathcal{S}_p(t) = \frac{1}{2} \langle p'_-(t)^2 \rangle^{-1}, \quad (9)$$

which in contrast to \mathcal{S}_c can be arbitrarily large [17]. The authors point out though that $\mathcal{S}_p \leq 1$ whenever two CV quantum systems can be represented by a positive P function (quantum optics notion of classicality), whereas the opposite would require collective squeezing.

C. Mutual information and other information-based correlations

Entropic measures are often used in different contexts to quantify the correlation between sub-parts. In many cases, these quantities have been compared to other classical synchronization measures. Here we briefly review the case where they have been proposed in relation to quantum synchronization.

Classical mutual information, associated to time series of system observables, has been used as a measure of classical synchronization [6]. The quantum mutual information (MI) of a whole density matrix ρ_{AB} is defined as

$$I(\rho) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \quad (10)$$

where ρ_A (ρ_B) is the reduced density matrix obtained tracing the subpart B (A) and where S stands for the von Neumann entropy: $S(\rho) = -\text{Tr}\{\rho \log \rho\}$. MI was proposed in Ref. [32] as synchronization witness. The authors considered two different models showing synchronization, that is, two coupled Van der Pol oscillators and two qubits inside optical cavities in the presence of driving. It was shown that the steady-state MI had the same qualitative behavior of, respectively, the complete synchronization measure of Eq. (5), and relative phase between the two qubits (measured with an indicator close to Pearson's parameter) during the transient. Comparisons between mutual information and synchronization had already been performed in Refs. [15, 16, 36] showing that in harmonic systems [15, 16] MI is more robust in the synchronization regime, while for spins coupled through the environment [36] it is not distinctive signature of synchronization.

In Refs. [15, 16, 32, 36], synchronization was also compared to quantum discord, the part of mutual information quantifying nonclassical correlations [45, 46], leading to similar results. Given a bipartite system AB , it is defined as the difference between $I(\rho)$ and the classical part of correlations $\mathcal{J}(\rho)_{\{\Pi_j^B\}} = S(\rho_A) - S(A|\{\Pi_j^B\})$, where the conditional entropy is $S(A|\{\Pi_j^B\}) = \sum_i p_i S(\varrho_{A|\Pi_i^B})$, $p_i = \text{Tr}_{AB}(\Pi_i^B \rho)$ and where $\varrho_{A|\Pi_i^B} = \Pi_i^B \rho \Pi_i^B / p_i$ is the density matrix after an optimal, complete projective measurement ($\{\Pi_j^B\}$) has been performed on B.

Generalized versions of MI can be obtained using the Rényi entropy $S_\alpha(\rho) = (1 - \alpha)^{-1} \log \text{Tr}\{\rho^\alpha\}$ (which reduces to S in the limit of $\alpha \rightarrow 1$). The Rényi-2 mutual information ($I_2(\rho) = S_2(\rho_A) + S_2(\rho_B) - S_2(\rho_{AB})$) was used by Bastidas *et al.* to detect chimera-type synchronization in a quantum network of coupled Van der Pol oscillators [47]. Chimera states describe the coexistence of synchronize and unsynchronized components [48].

Entanglement has also been considered in the context of synchronization. Lee and coworkers, studying the case of two dissipatively coupled Van der Pol oscillators, argued that the steady-state exhibits an entanglement tongue, the quantum analogue of the Arnold tongue [30]. Entanglement, after a truncation of the total Hilbert space of the two oscillators, was quantified using the concurrence E [49]. The concurrence between a pair of qubits, whose density matrix is ρ , is defined as $E = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$, where λ_r is the square root of the r th eigenvalue of $R = \rho \tilde{\rho}$ in descending order. Here, we have introduced $\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$, where ρ^* is the complex conjugate of ρ .

The linear entropy of the sub-part i $S(\rho_i) = 1 - \text{Tr}\{\rho_i^2\}$ can be used as entanglement quantifier provided that the whole state is pure. It was put in relation with synchronization of chimera states by Vienne and Aubourg in Ref. [37] in the case of a closed spin chain.

D. Correlations of observables

In Ref. [50] the synchronization between coupled non-linear cavities (a and b) was addressed considering normalized intensity correlations $g_2(a, b) = \langle n_a n_b \rangle / (\langle n_a \rangle \langle n_b \rangle)$ between the cavities (first and second harmonic) modes. The average $\langle \dots \rangle$ was temporal in the classic limit (neglecting quantum noise and considering classical trajectories), and it was in this limit that synchronization was addressed. In the quantum regime, the expectation value over the quantum (steady) state was considered so that g_2 is a measure of intensity correlations (capturing bunching/antibunching effects between the coupled systems). The transition between classical and quantum regime was described but addressing synchronization only in the classical regime and comparing it with steady state correlations when moving into the quantum regime.

The average of the collective operator

$$Z = \langle (\sigma_1^+ \sigma_2^- + \sigma_2^+ \sigma_1^-) \rangle \quad (11)$$

was used in Ref. [39] to detect the presence of phase locking between two (ensembles of) spins and then the synchronization between them. There, it was shown that decay rates of these correlations encode information about the spectral content of

the emitted radiation, which, in turn can be directly calculated using the two-time correlation function $Z = \langle (\sigma_1^+(\tau)\sigma_2^-(0) + \sigma_2^+(\tau)\sigma_1^-(0)) \rangle$, as already done in Ref. [38].

The value of spin-spin correlations $\langle \sigma_1^\alpha \sigma_2^\alpha \rangle - \langle \sigma_1^\alpha \rangle \langle \sigma_2^\alpha \rangle$ ($\alpha = x, y, z$) was also used by Hush *et al.* [42] as a sufficient criterion to assess the synchronization between the motion of two trapped ions. In such set-up, the spins represent the electronic degrees of freedom of the ions, and the value of their correlations was shown to be related to the relative phase distribution of the density matrix of the two motional degrees of freedom.

Looking at quantum correlations between observables to assess quantum synchronization is a natural strategy but it does not always capture the emergence of similar time evolutions between different sub-systems. This approach is actually often considered looking at stationary states loosing any relation with the classical counterpart of synchronization.

E. Kuramoto models

The Kuramoto model [51] was introduced by Yoshiki Kuramoto to study the synchronous behavior of a large set of coupled oscillators, which appears naturally in the context of chemical/biological systems. The equations of motion for the phase variable of the N oscillators take the form:

$$\dot{\theta}_i = \omega_i + \xi_i + \frac{K}{N} \sum_j \sin(\theta_i - \theta_j), \quad (12)$$

where ω_i are oscillator frequencies, ξ_i noise terms, and K quantifies the coupling. An order parameter is defined assuming mean-field coupling

$$r e^{i\psi} = \frac{1}{N} \sum_i e^{i\theta_i}, \quad (13)$$

where r represents the phase-coherence of the population of oscillators. Moving to a rotating frame where $\psi = 0$, the equations of motion become (without noise)

$$\dot{\theta}_i = \omega_i - K r \sin(\theta_i), \quad (14)$$

which is just a particle in a washboard potential. This means that whenever $|\omega_i| < Kr$ the phase is trapped and we have synchronization, otherwise the phase slips down the washboard. If the distribution of frequencies $g(\omega)$ is unimodal and centered around Ω , the phase transition to complete synchronization occurs for the critical coupling value $K_c = 2/\pi g(\Omega)$.

This model and similar ones have spurred a vast amount of research reviewed in [52]. Here we note that whenever a system can be reduced to a set of equations similar to the Kuramoto model, an analogous argument can be made to check whether there is phase locking or not. This is the case for coupled optomechanical oscillators as reported in Ref. [13]. The authors are able to reduce the mean field dynamics of the optomechanical array with all-to-all couplings (after carefully eliminating the amplitudes from the dynamics) to a Kuramoto-type model, which for only two units simplifies to

$$\delta\dot{\theta} = -\delta\Omega - C \cos(\delta\theta) - K \sin(2\delta\theta), \quad (15)$$

with $\delta\theta = \theta_2 - \theta_1$ and $\delta\Omega$ the frequency difference. Once here, pure analysis of the parameters in the equation directly point to either phase locking or phase slip.

Notably, translation of the Kuramoto model to the semiclassical domain was recently achieved [53] and it was shown that the picture of the washboard potential is a good intuitive guide, where now quantum tunneling can allow the phase to tunnel through maxima of the (otherwise phase-locking inducing) potential. This leads to the necessity of a higher critical coupling constant K_c in order to pin down and lock the phases in the model.

F. Other approaches

Quantum synchronization has also been addressed considering phase space representations. In Ref.[23] the collective phase-coherence among the components of an optomechanical array was characterized through an order parameter, Eq.(13), and looking at the transition of the Wigner representation from an angular-symmetric distribution to a coherent displaced state with time dependent phase. Under the assumption of infinite non-linearity, Van der Pol oscillators in presence of driving were considered in [10, 31] in strongly non-classical regimes, where the dynamics is governed by few Fock states: the authors discuss the similarity of the Wigner distribution with limit cycle appearing in the classical regime and the break of rotational invariance of W to characterize phase locking. The marginals of the relative phase are also a signature of phase locking as considered in Ref.[42]. Interestingly, phase state tomography has been recently reported in order to characterize phase diffusion and locking for an on-fiber optomechanical cavity operating into the classical regime [54]. Still, when writing this Chapter, there are not experimental results on synchronization reported in the quantum regime.

IV. SYNCHRONIZATION OF OSCILLATORS

A. Linear networks of Brwonian oscillators

Coupled harmonic oscillators dissipating in a bosonic environment represent the simplest setup where synchronization of quantum systems can occur [55]. Their intrinsic linearity precludes the possibility of synchronous limit cycles, but not the presence of a long transient where synchronization is present and also of steady oscillating states protected from dissipation. They have been studied for example in [15, 16, 34], where it was shown that eigenmodes in the system of coupled oscillators can display very different dissipation timescales for some parameter ranges. If one of the eigenmodes' dissipative rate is much smaller than that of the rest, this eigenmode dominates the dynamics of the coupled oscillators, and hence they become synchronized at the eigenfrequency of that mode. This synchronization is temporary but can be very long if such rate is small [15]. For certain situations of high symmetry, one of the eigenmodes of the coupled oscillators system can be isolated from the environment and then synchronization can last forever [16]. Situations of higher symmetry can also occur where more than one eigenmode is not dissipating, whereby synchronization can not happen [34]. More elaborate situations have been studied, for example in harmonic networks [16] it has been shown that the full network or only a motif inside it can be synchronized by tuning one of the frequencies.

One of the main conclusions in this type of systems is that dissipation induces quantum synchronization through diffusive coupling: be it for some parameter region or another, synchronization can always be achieved, with the sole exception of the separate baths case, where each coupled oscillator is attached to its own independent heat bath. It is easy to demonstrate mathematically that in this case, also eigenmodes are dissipating into equivalent independent heat baths, and therefore their dissipative rates are of equal size: no eigenmode then survives longer than the rest and thus synchronization is avoided. Different dissipation mechanisms that can arise in extended environments [56] lead to specific forms of diffusive couplings that can induce synchronization.

The Hamiltonian that describes this dynamics is

$$H = \sum_j \frac{p_j^2}{2m_j} + m_j \frac{\omega_j^2}{2} x_j^2 + \sum_{i \neq j} \lambda_{ij} (x_i - x_j)^2, \quad (16)$$

whereas dissipation is introduced by coupling each oscillator to its own environment -eventually with different dissipation strengths-, to a common one -with a homogeneous coupling or not-, or other combinations. For ease of exposition here we will present the commonly denominated 'common bath' case with system-bath interaction

$$H_{diss.} = \sum_j x_j \sum_k c_k Q_k, \quad (17)$$

where all system components couple equally to the same environment. In the case of two units with equal mass and different frequency the system reads

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{\omega_1^2}{2} x_1^2 + \frac{\omega_2^2}{2} x_2^2 + \lambda x_1 x_2, \quad (18)$$

where we have already absorbed quadratic contributions from the spring coupling into the oscillators frequencies. The oscillators in the bath are uncoupled among them and have frequencies ω_j , which together with the coupling constants c_k define in the continuum limit what is usually called the spectral density $J(\omega) = \sum_k \frac{c_k^2}{\omega_k} \delta(\omega - \omega_k)$. This function encodes basically all information about the dissipation characteristics and is usually taken to be 'Ohmic' [that is, $J(\omega) \sim \omega$] which gives a damping of the oscillators which is just proportional to their velocity. The linearity of the dynamics means that if we use an initial state which is Gaussian (all its information can be described from first and second moments), it will remain so at all times. Examples of Gaussian states [57, 58] widely used and relevant are displaced, squeezed and thermal states.

Synchronization can for example be displayed by the time evolution of second moments of quantum states, such as vacuum squeezed states, with the first moments being zero at all times. An example of this can be seen in Fig. 2, where the Pearson indicator is drawn for different detunings and coupling strength [15]. The shape is reminiscent of an Arnold tongue in classical physics, and has been also later observed with Van der Pol oscillators [32, 50]. The fact that first moments are zero while second moments are oscillating and synchronize does not seem to fit well with the synchronization error indicator, which in fact grows in the regions where the Pearson indicator clearly points to worse values.

Furthermore, information-based correlations are preserved through the common eigenmode that does not dissipate, and thus survive longer in the synchronized case. However, correlations cannot assess synchronization if they were not already present at some initial time and thus are neither good synchronization measures for this example. We note that the considered system focuses on the effect of dissipative coupling in harmonic arrays, and this analysis could be applied -for instance- the noisy precursor of an optomechanical system below the oscillation threshold.

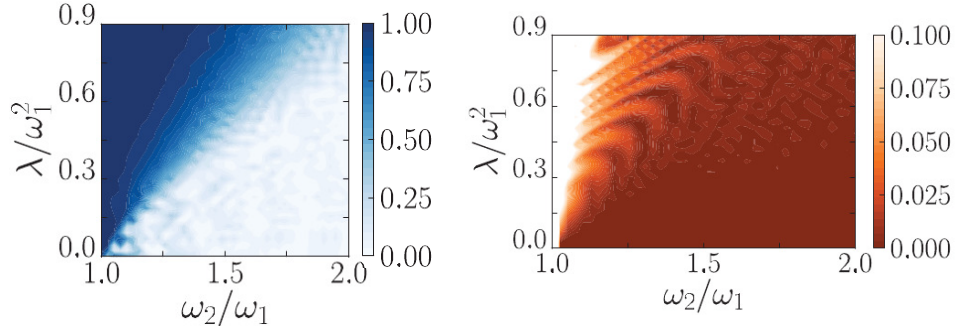


FIG. 2. Left: Pearson indicator of synchronization for different detunings and coupling strengths in the two coupled oscillators model dissipating into a common bath. They are initialized to squeezed vacuum states with zero first order moments. Synchronization is measured in the quadratures of each oscillators, i.e. $C_{\langle x_1^2(t) \rangle \langle x_2^2(t) \rangle}$ is plotted. Right: Quantum discord at the same value of time. A similar, although narrower, “tongue” is observed.

B. Self-sustained optomechanical oscillators

An important example of synchronization dynamics are optomechanical systems: capable of displaying limit cycles of constant amplitude, they provide an intuitive connection to what is known in the classical realm where the Kuramoto model is paradigmatic. Their ability to synchronize was first proven in [13], where it was shown that mechanically coupled optomechanical oscillators above their dynamical Hopf bifurcation can be described with a Kuramoto-type model; phase and anti-phase synchronization are displayed. The analysis in the case of a common bosonic mode [14] and experimental demonstration of the classical synchronization of such system [24] followed shortly. Quantum synchronization has been later reported: above a threshold mechanical coupling between optomechanical units, a regime of quantum phase-coherent mechanical oscillations arises [23]. The measure of quantum synchronization analogue to synchronization error was considered for optomechanical oscillators in [17].

Optomechanical systems [22] comprise a mechanical mode and a confined optical mode which is typically driven by a laser field. These modes are coupled nonlinearly through radiation pressure, provided by a movable mirror or a structure which can be deformed by the action of light such as a dielectric medium. The combination of an external pump and a nonlinear coupling provides stable limit cycles upon which synchronization can occur. The usual form of the Hamiltonian describing such dynamics is

$$H = \Delta a^\dagger a + \omega b^\dagger b + g a^\dagger a (b + b^\dagger) + iE(a - a^\dagger) \quad (19)$$

with $a(a^\dagger)$ the annihilation (creation) operators of light in the cavity, $b(b^\dagger)$ those of the mechanical mode and E is the intensity of the laser input. The Hamiltonian is written in the frame rotating with the laser frequency ω_L , which is detuned by $\Delta = \omega_C - \omega_L$ with respect to the cavity frequency ω_C . Both modes are further coupled to noise sources with strength κ, γ respectively, providing dissipation. Below a given threshold intensity of the laser, the amplitude of light and mechanical modes simply decay to the value enforced by the noise sources. However, above that intensity, multistability and limit cycles of increasing complexity can be observed. The final ingredient to observe synchronization is coupling, via the mechanical or optical degrees of freedom, several optomechanical systems. For illustrative purposes we choose here mechanical coupling of the type $H_{int} = \mu(b_1 b_2^\dagger + b_1^\dagger b_2)$. We will follow here the example of dynamics given in [17] to compare different measures of synchronization and correlations. After a transient, limit cycles are achieved for each degree of freedom and we can linearize them around their stable orbits with the usual ansatz: $\hat{a}_j = A_j(t) + \delta \hat{a}_j$ and $\hat{b}_j = B_j(t) + \delta \hat{b}_j$, with capital letters representing the limit cycles as a classical variable, and $\delta \cdot$ the linear displacements (fluctuations) with respect to them. The ansatz is then used to neglect nonlinear terms in the dynamics. Finally, the fluctuations can be arranged in the form of a covariance matrix whose time evolution can be integrated, yielding both the dynamical content and the quantum/classical information content.

Coupled optomechanical oscillators represent a good platform combining sufficiently complex dynamics and the possibility to assess different informational measures, so that proposed indicators of synchronization can be compared, as seen in Fig. 3. We consider two optomechanical systems like in Ref. [17]. The qualitative behaviour is similar for all indicators: an initial build up and a final stage of full synchronization by a stationary value of all quantities.

As mentioned before, perfect synchronization can be identified looking at the absolute value of the indicator only for the Pearson coefficient (Sect. III A), which really shows perfect synchronization (value ~ 1) for first and second moments of the mechanical observables (Fig. 3b, time $\gtrsim 500$). The synchronization error indicators have a rather low value, Fig. 3a, compared to their maximum attainable 1 (see Sect. III B), and we have checked that for different initial conditions of the mechanical first moments, it can be increased, with very similar qualitative behaviour. The possibility to reach its maximum bound Eq.

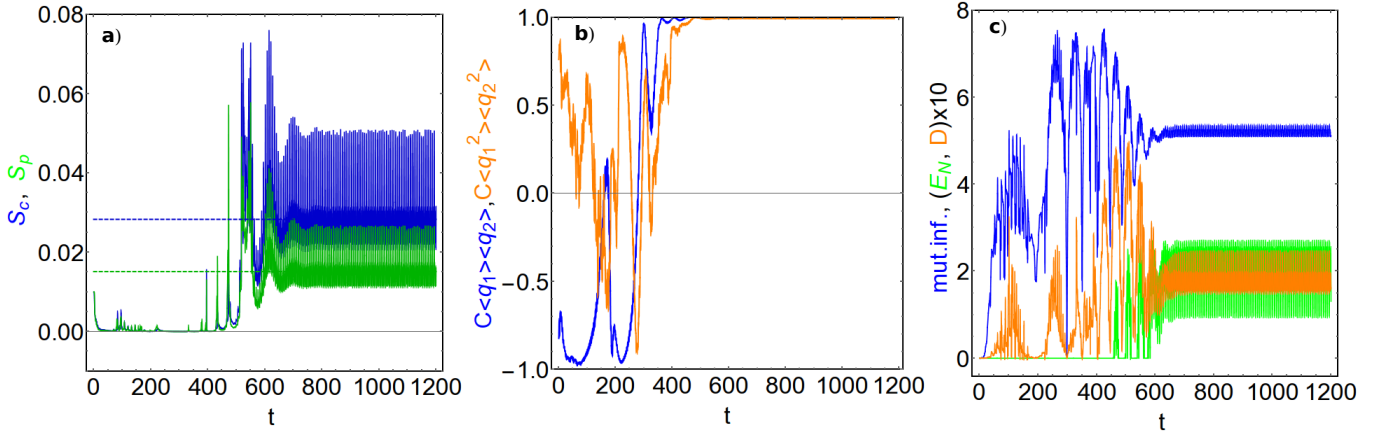


FIG. 3. Comparison of several synchronization measures: (left) error synchronization S_c, S_p , (middle) Pearson indicator for first and second momenta $C_{\langle q_1 \rangle \langle q_2 \rangle}, C_{\langle q_1^2 \rangle \langle q_2^2 \rangle}$, and (right) correlations: mutual information, logarithmic negativity and quantum discord. We set parameters $\omega_1 = 1, \omega_2 = 1.005, g = 0.005, \mu = 0.02$, detunings $\Delta_j = \omega_j, \kappa = 0.15, \gamma = 0.005$ and laser input $E = 320\kappa = 48$, as in Ref. [17]. The initial condition for first moments are $\langle q_1(0) \rangle = 100$ and $\langle q_2(0) \rangle = -100$, and all other first moments zero. The second order moments at $t = 0$ are 100 times their vacuum value. Changing these initial conditions do not change qualitatively the results.

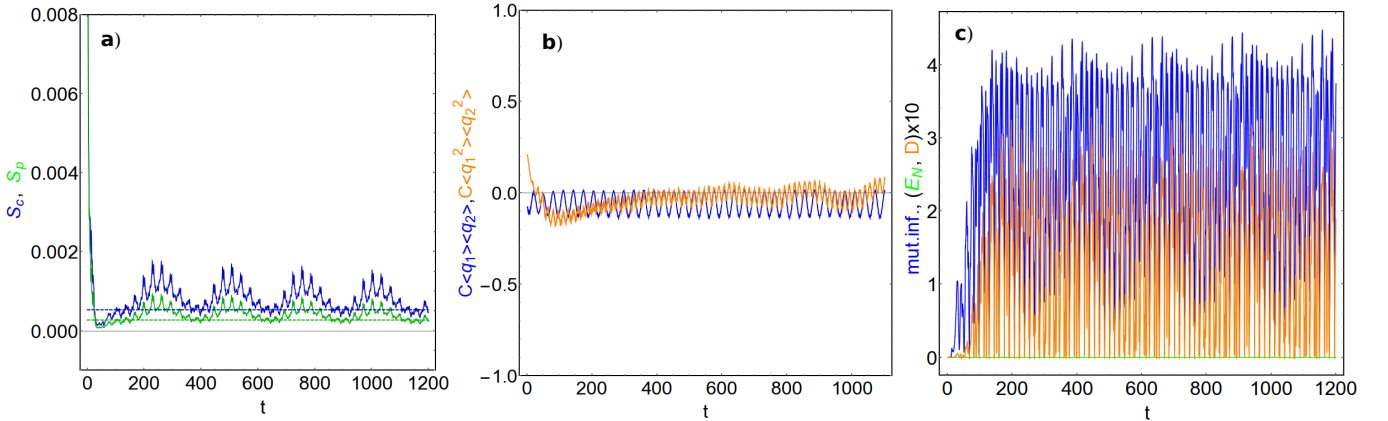


FIG. 4. Same parameters as Fig. 3, but with higher detuning $\omega_2 = 1.2$, so there is no synchronization. Notice that the scale in (a) is ten-fold lower than in Fig. 3a.

(6) in such system has not been reported, although for fixed initial conditions it might be informative to assess the quality of synchronization when changing Hamiltonian parameters (as when comparing Fig. 3 and Fig. 4). As a note, while these different indicators can signal stable synchronization for $t > 600$, the Pearson coefficient spot it sooner. Comparing figures 3c and 4c we can conclude too that any measure based on classical/quantum correlations (of information-theoretic character) is not of too much value if we compare absolute values. Introducing initial squeezing in the mechanical modes changes the time profile of all correlations, but as expected not their final stable values.

Finally, what is important to note is that the stability in time of the indicators is a necessary condition for signaling synchronization, and once that is achieved, comparison of absolute values might yield some extra information, although only the Pearson indicator is a bona fide absolute measure in this sense.

V. SYNCHRONIZATION OF INTERACTING SPINS

When considering precessing spins, the behavior of local spectra can be used to extract direct information about the presence of a synchronized dynamics. This method was adopted by Orth and coworkers in Ref. [12], where they considered two interacting spins dissipating through a common environment, and observed that there is a regime in the parameter space of the system where only a single frequency appears in the spectrum of both the local observables. This single-line spectrum is not the only possible manifestation of synchronization. It is indeed possible, like in the infinite-dimension Hilbert space cases discussed above, that

the collective dynamics favours the suppression of some spectral lines, while one of them has a very long life-time, leaving the system synchronized during the transient decay leading to steady state [36, 44]. In these cases, the Pearson's measure can be adopted to observe the dynamical setting-up of synchronization. It can be defined considering the expectation values of any arbitrary operator for each spin A_k , $k = 1, 2$, decomposed in the single-spin basis $\{\sigma_x^k, \sigma_y^k, \sigma_z^k, I_d^k\}$:

$$A_k = a_x^k \sigma_x^k + a_y^k \sigma_y^k + a_z^k \sigma_z^k + a_d I_d^k. \quad (20)$$

Actually, in many cases, it is enough to consider one spin direction. For instance, in Ref. [36], the z components of the two spins were synchronized from the beginning and the interesting part of the dynamics concerned the $x - y$ plane.

In the following, we are going to compare the Pearson's measure of quantum synchronization with correlation indicators in a model of two detuned spins interacting through an Ising-like coupling:

$$H_S = \frac{\omega_1}{2} \sigma_1^z + \frac{\omega_2}{2} \sigma_2^z + \lambda \sigma_1^x \sigma_2^x. \quad (21)$$

Let us assume that the spins experience a dissipative dynamics induced by the presence of a thermal environment weakly coupled to the system through

$$H_I = \sum_k g_k (a_k^\dagger + a_k) (A \sigma_1^x + \sigma_2^x). \quad (22)$$

Here, the annihilation (creation) operators a_k (a_k^\dagger) act on the bath degrees of freedom and the coefficient A determines the ratio between the strength of the two spin-bath couplings. For the sake of clarity, in the following we will only discuss the two extreme cases $A = 0$ (local environment) and $A = 1$ (common environment). In order to derive a master equation describing the dynamics of the two spins, it is necessary to know the diagonal form of H_S , which can be obtained applying the standard Jordan-Wigner transformation, mapping spins into spinless fermions, defined as $\sigma_1^z = 1 - 2c_1^\dagger c_1$, $\sigma_2^z = 1 - 2c_2^\dagger c_2$, $\sigma_1^x = c_1^\dagger + c_1$, $\sigma_2^x = (1 - 2c_1^\dagger c_1)(c_2^\dagger + c_2)$ [59]. We have

$$H_S = E_1 (\eta_1^\dagger \eta_1 - 1/2) + E_2 (\eta_2^\dagger \eta_2 - 1/2), \quad (23)$$

with $E_1 = \frac{1}{2} \left(\sqrt{4\lambda^2 + \omega_+^2} + \sqrt{4\lambda^2 + \omega_-^2} \right)$ and $E_2 = \frac{1}{2} \left(\sqrt{4\lambda^2 + \omega_+^2} - \sqrt{4\lambda^2 + \omega_-^2} \right)$ where $\omega_\pm = \omega_1 \pm \omega_2$. The quasi-particle fermion operators are obtained combining the Bogoliubov transformation $c_1 = \cos \theta_+ \xi_1 + \sin \theta_+ \xi_2^\dagger$, $c_2 = \cos \theta_+ \xi_2 - \sin \theta_+ \xi_1^\dagger$ together with the rotation $\xi_1 = \cos \theta_- \eta_1^\dagger + \sin \theta_- \eta_2^\dagger$, $\xi_2 = \cos \theta_- \eta_2^\dagger - \sin \theta_- \eta_1^\dagger$.

The spectral density $J(\omega) = \sum_k g_k^2 \delta(\omega - \Omega_k)$ is assumed to follow, apart from a high-frequency cut-off, the Ohmic power law $J(\omega) \sim \omega$. Assuming weak dissipation, the qubit pair dynamics can be studied in the Born-Markov and secular approximations [60] with Lindblad master equation $\dot{\rho}(t) = -i[H_S + H_{LS}, \rho(t)] + \mathcal{D}[\rho(t)]$, where the Lamb shift H_{LS} commutes with H_S and where $\mathcal{D}[\rho(t)]$, which takes into account dissipation, is the sum of four terms, each of them associated to one of the four transition frequencies $\pm E_i$ ($i = 1, 2$):

$$\mathcal{D}(\rho) = \sum_{i=1}^2 \tilde{\gamma}_i^+ \mathcal{L}[\eta_i](\rho) + \sum_{i=1}^2 \tilde{\gamma}_i^- \mathcal{L}[\eta_i^\dagger](\rho), \quad (24)$$

Here, the Lindblad superoperators are defined as $\mathcal{L}[\hat{X}](\rho) = \hat{X}\rho\hat{X}^\dagger - \{\hat{\rho}, \hat{X}^\dagger\hat{X}\}/2$. The exact value of the decay rates γ_i^\pm will depend on the nature of the system-bath coupling. In the case of local dissipation $A = 0$, their specific form can be found in Ref. [44]. As already discussed, synchronization takes place if there is substantial separation between the two largest γ 's determining the dynamics. In this case, local degrees of freedom undergo quasi-monochromatic oscillations and their relative phases get locked.

A. Spin synchronization

As discussed in Ref. [44], the case of a local environment $A = 0$ shows an “anomalous” synchronization pattern. Indeed, unlike classical manifestations of synchronization and quantum synchronization induced by a common environment (Refs. [15, 16, 36]), it is greatly enhanced in the strong detuning $\Delta = |\omega_1 - \omega_2|$ regime, while the direct spin-spin coupling λ has a partially detrimental effect. Turning attention to the case $A = 1$, the presence of a common bath has the tendency to facilitate spontaneous synchronization. In order to observe it, it is fundamental for the two spins to have an interaction strong enough as to compensate the detuning. Furthermore, in the local-bath case, depending on the Hamiltonian parameters, synchronization

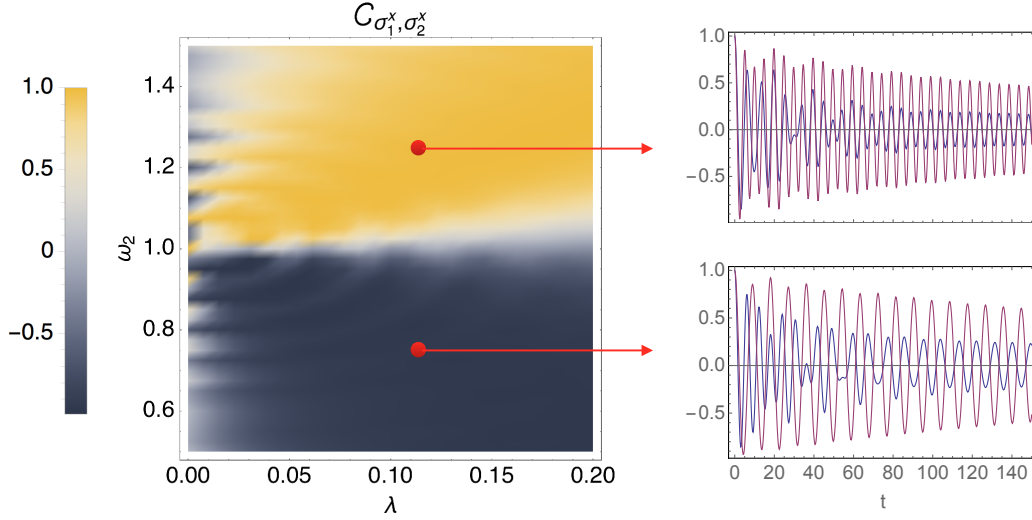


FIG. 5. Left panel: synchronization diagram as a function of ω_2 and λ for a local bath. $C_{\sigma_1^x, \sigma_2^x}$ has been calculated at time $t = 75$ (in units of ω_1) using a time window of $\tau = 10$. As explained in the text, the local bath ($A = 0$) is assumed to be Ohmic with cut-off frequency $\omega_c = 20$, and its temperature is $T = 0$. The initial state is $|\psi(0)\rangle = (|\uparrow\rangle + |\downarrow\rangle)(|\uparrow\rangle + |\downarrow\rangle)/2$. Left panels: $\sigma_1^x(t)$ (blue) and $\sigma_2^x(t)$ (red) assuming $\omega_2 = 1.25$ and $\lambda = 0.11$ (top) and $\omega_2 = 0.75$ and $\lambda = 0.11$ (bottom).

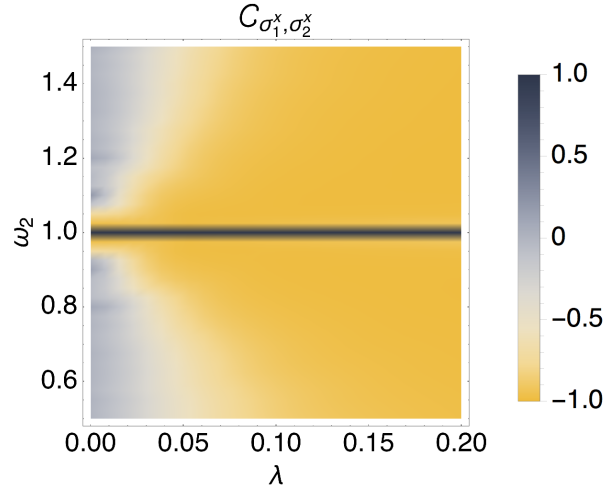


FIG. 6. $C_{\sigma_1^x, \sigma_2^x}(t = 75, \tau = 10)$ as a function of ω_2 and λ in the presence of a common bath ($A = 1$). All other conditions as in Fig. 5.

can appear both in phase and in anti-phase. This feature is suppressed in the presence of a common environment, where only anti-synchronization can be observed. This is due to the different interplay between the γ 's in the two scenarios.

The anomalous synchronization emerging from a local environment is shown in the $\{\Delta - \lambda\}$ diagram of Fig. 5. The local observable used to calculate the Pearson's parameter \mathcal{C} are, respectively, σ_1^x and σ_2^x . even if the calculation could be extended to generic local operators without qualitative changes in the results. In the left panel, we show the synchronization diagram, showing the transition from phase to anti-phase, while in the right panels we plot the trajectories of the two local observables in the two distinct regimes.

On the other hand, the behavior of \mathcal{C} for a common bath is displayed in Fig. 6 and it is very much similar to the characteristic Arnold tongues emerging in classical synchronization problems. In this case, anti-synchronization emerges provided that the spin-spin coupling is not too small with respect to the detuning. As a singular behavior, around $\Delta = 0$, the system shows "trivial" synchronization, given that the two spins become indistinguishable.

The complementarity and the qualitative difference of the synchronization diagrams emerging in the two cases under study will be used in the following of this section to compare the Pearson's measure to correlation quantifier, namely, spin-spin correlations $\langle \sigma_i^+ \sigma_j^- \rangle$, mutual information, and entanglement.

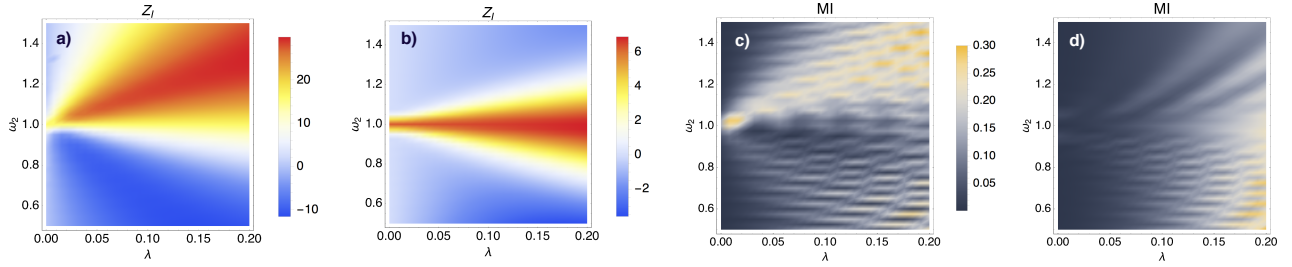


FIG. 7. Z_I (a,b) and mutual information MI (c,d) as a function of ω_2 and λ for a local bath $A = 0$ (a,c) and for a common bath $A = 1$ (b,d). All parameters as in Fig. 5.

B. Spin correlations

As noticed in Ref. [39], $Z = \langle \sigma_1^+ \sigma_2^- + \sigma_2^+ \sigma_1^- \rangle$ plays the role of a phase locking indicator when applied to interacting spins, and then can be used as a synchronization measure (see Sec. III D). At a first sight, \mathcal{C} and Z evolve independently, as the sets of equations of motion of their respective matrix elements are not coupled to each other. Actually, the constraints, to which a physical density matrix is subject to, make the behavior of the two indicators very close to each other. This aspect is discussed in great detail in Ref. [61], in the case of independent spins, where the interplay between spontaneous synchronization and superradiance is studied. In fact, in order for spontaneous synchronization to emerge, the whole system needs to support a long-lasting collective mode, which unavoidably displays spin-spin correlations.

These qualitative considerations are confirmed in the cases we are discussing here: in both models (of local and global dissipation), low-quality synchronization \mathcal{C} is always accompanied by a value for Z close to zero. On the other hand, within the synchronized regions, Z is significantly enhanced. Furthermore, the sign of Z is reminiscent of the phase-anti-phase form of synchronization. These results are shown in Fig. 7, where, in order to wash out faster oscillations and regularize the picture, the time integral of Z ($Z_I = \int_{t=0}^{100} Z(t') dt'$) is plotted as a function of ω_2 and λ . Both cases of local ($A = 0$) and global ($A = 1$) dissipation are shown. For a local bath, Fig. 7a, the change from positive to negative values for the spin-spin correlation parameter takes place in the same region where \mathcal{C} passes from synchronization to anti-synchronization (compare with Fig. 5 a,b). In view of the previous considerations, this change is suppressed for $A = 1$, Fig. 7b. It is worth noticing that the "anomalous" synchronization peak around $\omega_1 = \omega_2$ displayed by \mathcal{C} (Fig. 6) is broadened by Z_I (Fig. 7b), whose value appears smoother at critical changes.

C. Mutual information and entanglement

Mutual information (MI) as a quantum synchronization witness in spin systems was proposed by Ameri *et al.* in a work dealing with a system of two qubits placed in two coupled cavities where only the first one is driven by a laser, while the second one is populated by the photons leaking from the first cavity [32]. In that example, it was shown that the steady-state mutual information was reminiscent of the synchronized oscillations of local operators in the pre-steady-state regime.

In the following we show that, in the models we are investigating, MI does not play the witnessing role suggested in Ref. [32]. As a first observation, we notice that we deal with systems decaying towards an equilibrium state (the Gibbs state) that only depends on the Hamiltonian parameters, while the synchronization diagram depends critically on the properties of the environment. To make this point clearer, both cases we are considering here admit the same equilibrium state, whilst the two synchronization diagrams are radically different. One may ask if some information about synchronization appears in the dynamical behavior of MI instead of its asymptotic value. For this reason, we considered the time at which synchronization starts to be solid ($t = 80$ in Fig. 7 c,d) and calculated MI for the two models (at longer times MI would rapidly converge to zero everywhere). The two behaviors indicate a very weak connection between MI and spontaneous synchronization. Starting from a factorized state, the coupling λ immediately produces a quite robust amount of MI (depending on the strength of λ) between the two spins. Then, the presence of dissipation makes this correlation disappear, but it seems that the way MI fades away is not

connected with the building up of a synchronized dynamics. A very similar argument can be applied to entanglement, that can be quantified using the concurrence E [49]. The dynamical behavior of E (not shown) displays qualitative features very close to the ones on MI .

Besides the specific models studied here, the argument can be made more general considering the case of a purely dephasing dynamics. On the one hand, as discussed in Ref. [36], such a process is not able to induce any synchronization, as no time scale separation takes place. On the other hand, entanglement and mutual information can converge to a finite value under the same circumstances.

VI. DISCUSSION AND CONCLUSIONS

Present research on quantum synchronization has just started to unveil the distinctive features of this phenomenon. Several factors, known to influence it in the classical regime [1] are under study, including, for instance, non-linearity, dissipation, noise, forcing, mutual or directional coupling between inhomogeneous components, or time delay. Experiments reporting distinctive signatures of quantum synchronization are expected to flourish in the next years.

The question about what is essential of quantum synchronization with respect to the classical one is intimately related to the interplay between temporal and quantum correlations. From the previous analysis we can establish few relevant criteria to approach the phenomenon of quantum synchronization as described by different measures and to assess usefulness and meaningfulness in each specific context:

- **Absolute reference value:** In order to be able to assess the amount of synchronization in different regimes, it is important for a measure to be bounded and to have a definite value associated to the perfect emergence of full synchronization. The Pearson's parameter Eq. (3) reaches values very close to the maximum attainable $|\mathcal{C}_{A_1, A_2}(t|\Delta t)| \simeq 1$ whenever good synchronization emerges. Similarly, the synchronization error Eq. (5) is bounded in the quantum case, whereas the classical one is not. Still, reported values are rather modest in the case of coupled optomechanical oscillators [17] and saturation of this bound that would correspond to the best quantum synchronization has not yet been reported.
- **Time dynamics dependence:** The concept of synchronization is relative to the time evolution of system's observable or variables and a measure of synchronization should reflect it covering a time window of the system dynamics, like in temporal averages for instance, or being robust during evolution. This is the case for several measures in different ways: some are based on time averages (e.g. Pearson's parameter (3)), others maintain distinct higher values during synchronization (e.g. synchronization error (5)), and others assess the time stability of the process (as Lyapunov exponents [62]). In general, the problem when looking at instantaneous (quantum) correlations, MI etc. is that they can be instantaneously huge even when there is no synchronization, as shown in Fig. 4. On the other hand, looking at asymptotic values is not always leading to an insightful synchronization condition.
- **Local vs. non-local** Among the reported measures of synchronization, some refer to local observables of the synchronized systems (like Pearson factors, or local phases in Kuramoto models) while other refer to quantum correlations present in the composed system (in this sense being 'non-local'). The possibility to associate a genuine quantum correlation to synchronization is clearly appealing to distinguish it from classical synchronization, as for instance with the synchronization error [17]. On the other hand, this can give rise to spurious definitions of quantum synchronization, actually not related at all with this dynamical phenomenon. This question is still open and few further considerations are given below.

In the attempt to identify a measure for a genuine quantum synchronization, different indicators have been proposed that actually do not refer to observables but to the full quantum state, as discussed in Sec. III. Invoking generic quantum correlation as a measure of synchronization is in general not convincing. As an example, a bipartite Bell-state is strongly correlated under any possible definition of correlation, but in general this has nothing to do with synchronization. As a matter of fact, any local unitary would leave it unchanged, while altering the dynamics of the components of the system can alter dynamical synchronization. Even if non-local correlations are not necessarily associated to specific synchronization phenomena, it is important to remind that quantum correlations and dynamical synchronization can occur under the same conditions in some systems [15–17, 30]. Still, quantum correlations that capture specific signatures of synchronization are not generally established. Looking at the examples we have treated here we can also draw some conclusions.

Optomechanical self-sustained oscillators can achieve synchronization and several parameters like the error and Pearson indicators give a similar insight to mutual information or other correlations (Fig. 3 and 4). Synchronization error however displays a small value far from the maximum bound not providing an absolute indication for the synchronization degree (Fig. 3a). Furthermore, correlations signal synchronization not by their value, but by having a final stable nonzero value, in contrast to a highly oscillating one in the case of no synchronization.

In the case of linear dissipative oscillators the situation worsens. The Pearson factor gives a valuable guide to look for synchronization, while all the other indicators fail: synchronization error does not provide a good estimate of the behaviour

of synchronization with respect to the system's parameters and other information measures are strongly initial state dependent. Still, robust quantum correlations (such as discord) can witness synchronization, as both emerge under the same circumstances in this case. At some level, all of the measures can be used to yield some insight, however they require some craftsmanship efforts as compared to Pearson coefficients.

The literature about quantum synchronization in spins is much more limited with respect to the case of harmonic oscillators or optomechanical systems. Furthermore, in many cases, synchronization has been assessed using ad-hoc witness measures more than quantifiers. This chapter represents the first attempt to compare such quantities. As a result, we observed consistent indication of synchronization between Pearson's and spin-spin Z indicators, due to a strong interplay between phase-locking dynamics and the dynamics of the local observables. In contrast, mutual information and entanglement fail to give any useful information.

Finally, all the previous measures of synchronization could be modified to account for more general forms of synchronization. In all the discussed cases, synchronization is either in-phase or anti-phase. It is worth remarking that, in general, delayed synchronization can also emerge and synchronization indicators need to be improved to catch this effect. This can be easily done, for instance in the case of the Pearson's parameter, allowing one of the two sliding windows to open at a time different from the other one, that is equivalent to delay the time of one of the observable expectation value

$$\mathcal{C}_{A_1(t), A_2(t+\tau)}(t|\Delta t). \quad (25)$$

Similarly this could be done for all indicators based on local observables. Another way of improving synchronization indicators consists in correcting possible relative amplitude mismatch effects, similarly to the conditional variance factor appearing in the context of the EPR correlations [63].

To conclude we would like to stress that the field is still in its early stages and this work is the first attempt to assess meaning and utility of different synchronization measures as well as their possible dependence to the specific features of the system under study. Up to now, no experimental results in the quantum domain are at hand. Therefore there is plenty of room for improvement and surprises, both regarding the theoretical framework and possible practical applications of potential use as quantum technologies.

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